Some things that we have, explained very briefly, with slightly different notation

We announce the setting for our axiom system by declaring our *preliminary assumptions* to be language, logic, set theory, and the real numbers.

The theory begins:

Undefined terms: $\mathscr{P}$, $\mathscr{L}$, d, m.

> **Axiom 1** *Incidence Axiom*
>
> a $\mathscr{P}$ and $\mathscr{L}$ are sets; an element of $\mathscr{L}$ is a subset of $\mathscr{P}$.
> b If P and Q are distinct elements of $\mathscr{P}$, then there is a unique element of $\mathscr{L}$ that contains both P and Q.
> c There exist three elements of $\mathscr{P}$ not all in any element of $\mathscr{L}$.

We are going to call the elements of $\mathscr{P}$ *points* and the elements of $\mathscr{L}$ *lines*. By (a) of the Incidence Axiom, we are taking the point of view that a line *is* a set of points. Thus, we automatically have an incidence relation for points and lines given by set membership. Because of (b), the Incidence Axiom might be called the *Straightedge Axiom*. We need (c) to get our *plane* off the ground, as without this there might be no points or lines at all or there might be just exactly one line.

> **Axiom 2** *Ruler Postulate* $d:\mathscr{P}\times\mathscr{P}\rightarrow \mathbf{R}$, $d:(P, Q)\leftrightarrow PQ$ is a mapping such that for each line l there exists a bijection $f:l\rightarrow\mathbf{R}$, $f:P\leftrightarrow f(P)$ where
>
> $$PQ=|f(Q)-f(P)|$$
>
> for all points P and Q on l.

DEFINITION 12.1 ***Pasch's Postulate*** or PASCH: If a line intersects a triangle not at a vertex, then the line intersects two sides of the triangle. ***Plane-Separation Postulate*** or PSP: For every line l there exist convex sets H_1 and H_2 whose union is the set of all points off l and such that if P and Q are two points with P in H_1 and Q in H_2 then $\overline{PQ}$ intersects l.

You should recognize that the following three statements are equivalent to PASCH: (1) If a line intersects the interior of a side of a triangle, then the line intersects another side of the triangle. (2) If a line intersects a triangle, then the line intersects two sides of the triangle. (3) If a line does not intersect either of two sides of a triangle, then the line does not intersect the third side of the triangle. Of course, a line may intersect all three sides of a triangle.

Axiom 3 *PSP* $\forall l \in \mathscr{L}$ $\exists$ convex sets H_1 and H_2 $\ni$

1 $\mathscr{P} \setminus l = H_1 \cup H_2$,
2 $P \in H_1, Q \in H_2, P \neq Q \Rightarrow \overline{PQ} \cap l \neq \varnothing$.

DEFINITION 12.3 The sets H_1 and H_2 in Axiom 3 are ***halfplanes*** of line l, and l is an ***edge*** of each halfplane. A halfplane of $\overleftrightarrow{AB}$ is a ***halfplane*** of $\overrightarrow{AB}$ and a ***halfplane*** of $\overline{AB}$.

We are ready for the statement of our next axiom, which determines some properties of the undefined term m.

Axiom 4 *Protractor Postulate* m is a mapping from the set of all angles into $\{x | x \in \mathbf{R}, 0 < x < \pi\}$ such that

a if $\overrightarrow{VA}$ is a ray on the edge of halfplane H, then for every r such that $0 < r < \pi$ there is exactly one ray $\overrightarrow{VP}$ with P in H such that $m\angle AVP = r$;

b if B is a point in the interior of $\angle AVC$, then $m\angle AVB + m\angle BVC = m\angle AVC$.

You should stop and examine the Protractor Postulate in detail. As well as deciding what the axiom says, you should think about what it does not say. How close does Axiom 4 come to incorporating all that you *see* when you look at a protractor?

DEFINITION 14.1 Mapping m is called the ***angle measure function.*** The ***measure*** of $\angle AVB$ is $m\angle AVB$. If an angle has measure $k\pi$, then the angle is said to be ***of*** 180k ***degrees.*** $\angle AVB \simeq \angle CWD$ iff $m\angle AVB = m\angle CWD$, in which case we say that $\angle AVB$ is ***congruent*** to $\angle CWD$.

Axiom 5 SAS Given $\triangle ABC$ and $\triangle DEF$,
if $\overline{AB} \simeq \overline{DE}$, $\angle A \simeq \angle D$, and $\overline{AC} \simeq \overline{DF}$,
then $\triangle BAC \cong \triangle EDF$.

Let's do it!

Axiom 6 HPP If point P is off line l, then there exist two lines through P that are parallel to l.

Our axiom system, now called the *Bolyai–Lobachevsky plane*, is as consistent as the Euclidean plane or the real numbers (Section 23.2).

Axiom 6, the Hyperbolic Parallel Postulate, could be weakened to require only the existence of nonincident point P_0 and line l_0 such that there exist two lines through P_0 that are parallel to l_0. This follows from Proposition Y of Theorem 23.7. On the other hand, Axiom 6 could be replaced by our next theorem.

13 The Measure of an Angle

As we have progressed through the first twelve sections, we have built up an axiom system piece by piece. Starting with the underlying set structure in an abstract geometry, we added the incidence axioms, the ruler postulate, and the plane separation axiom. So we can say that we had

Axiom 1: The Incidence Axiom

Axiom 2: The Ruler Postulate

Axiom 3: PSP

Now we want to add another axiom (Axiom 4: The Protractor Postulate) to make our geometry look more like the geometry that is studied in high school. In this section we shall define what is meant by an angle measure and indicate how angle measures are defined in our two basic models.

After we have an angle measure our geometry will look very much like "high school geometry." However, we will still be missing one important assumption. That assumption is the Side-Angle-Side (SAS) congruence axiom (Axiom 5: SAS). Without it, some results can occur which are very unusual for someone accustomed only to Euclidean geometry. In particular, we will see in examples that without SAS it is possible for the sum of the measures of the angles of a triangle to be greater than 180 degrees, or for the length of one side of a triangle to be longer than the sum of the other two.

Definition (angle measure or protractor) Let r_0 be a fixed positive real number. In a Pasch geometry, an angle measure (or protractor) based on r_0 is a function m from the set $\mathcal{A}$ of all angles to the set of real numbers such that

(i) If $\measuredangle ABC \in \mathcal{A}$ then $0 < m(\measuredangle ABC) < r_0$;

(ii) If $\overrightarrow{BC}$ lies in the edge of the half plane H_1 and if θ is a real number with $0 < \theta < r_0$, then there is a unique ray $\overrightarrow{BA}$ with $A \in H_1$ and $m(\measuredangle ABC) = \theta$;

(iii) If $D \in \operatorname{int}(\measuredangle ABC)$ then $m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC)$.

We should note that the definition of an angle measure does not even make sense unless we have PSA (or equivalently, Pasch) since we must use the idea of the interior of an angle. If $r_0 = 180$, m is called degree measure. If $r_0 = \pi$, then m is called radian measure. If $r_0 = 200$, then m is called grade measure. Traditionally, degree measure is used in geometry. Radian measure is used in calculus because it makes the differentiation formulas most natural.

Convention. Except in Section "Euclidean and Poincaré Angle Measure", we shall always use degree measure ($r_0 = 180$) without further assumption.

Definition (protractor geometry) A protractor geometry $\{\mathcal{S}, \mathcal{L}, d, m\}$ is a Pasch geometry $\{\mathcal{S}, \mathcal{L}, d\}$ together with an angle measure m.

Definition (Euclidean angle measure) In the Euclidean Plane, the Euclidean angle measure of $\measuredangle ABC$ is

$$m_E(\measuredangle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B\rangle}{\|A-B\| \cdot \|C-B\|}\right)$$

1. In $\mathcal{E}$ what is $m_E(\measuredangle ABC)$ if $A(0,3)$, $B(0,1)$ and $C = (\sqrt{3}, 2)$?

Proposition m_E is an angle measure on $(\mathbb{R}^2, \mathcal{L}_E, d_E)$.

Definition (Euclidean tangent, Euclidean tangent ray) If $\overrightarrow{BA}$ is a ray in the Poincaré Plane where $B(x_B, y_B)$ and $A(x_A, y_A)$, then the Euclidean tangent to $\overrightarrow{BA}$ at B is $T_{AB} =$

$$\begin{cases} (0, y_A - y_B), & \text{if } \overleftrightarrow{AB} \text{ is a type I line} \\ (y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B < x_A \\ -(y_B, c - x_B), & \text{if } \overleftrightarrow{AB} \text{ is a type II line } {}_cL_r,\ x_B > x_A. \end{cases}$$

The Euclidean tangent ray to $\overrightarrow{BA}$ is the Euclidean ray $\overrightarrow{BA'}$ where $A' = B + T_{BA}$.

Definition (Poincaré angle measure) The measure of the Poincaré angle $\measuredangle ABC$ in $\mathbb{H}$ is

$$m_H(\measuredangle ABC) = m_E(\measuredangle A'BC') = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\| \cdot \|T_{BC}\|}\right)$$

2. In the Poincaré Plane find the measure of $\measuredangle ABC$ where $A(0,1)$, $B(0,5)$, and $C(3,4)$.

Proposition m_H is an angle measure on $(\mathbb{H}, \mathcal{L}_H, d_H)$.

3. Let $A(0,1)$, $B(0,5)$, and $C(3,4)$ be points in the Poincaré Plane $\mathcal{H}$. Find the sum of the measures of the angles of $\triangle ABC$.

Convention. From now on the terms Euclidean Plane, Poincaré Plane, and Taxicab Plane will refer to the protractor geometries $\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E, m_E\}$, $\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H, m_H\}$, $\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T, m_E\}$.

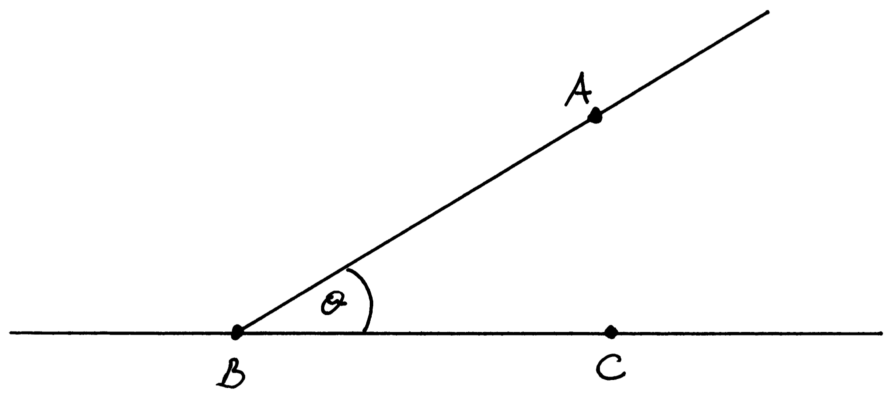
Mjeva ugla

Definicija (mjera ugla ili protractor)

Neka je r_0 fiksiran pozitivan realan broj. U Pasch-ovoj geometriji, mjera ugla (ili protraktor) baziran na r_0 je funkcija m sa skupa $\mathcal{A}$ svih uglova u skup realnih brojeva takva da

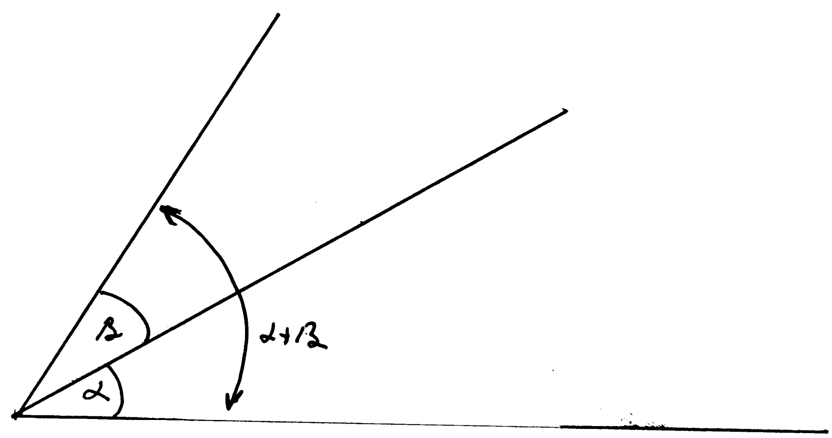
(i) Ako je $\measuredangle ABC \in \mathcal{A}$ tada $0 < m(\measuredangle ABC) < r_0$;

(ii) Ako $\overrightarrow{BC}$ pripada ivici poluravni H_1 i ako je θ realan broj sa $0 < \theta < r_0$, tada postoji jedinstvena polupravа $\overrightarrow{BA}$ sa osobinom $A \in H_1$ i $m(\measuredangle ABC) = \theta$



(iii) Ako je $D \in \operatorname{int}(\measuredangle ABC)$ tada

$$m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC)$$



Dogovor

Osim u lekciji "Euklidova i Poincaré-ova mjera ugla" uvijek ćemo koristiti mjeru u stepenima ($r_0 = 180$) bez daljih pretpostavki.

Definicija (protractor geometrija)

Protraktor geometrija $\{\mathcal{S}, \mathcal{L}, d, m\}$ je Pašova geometrija $\{\mathcal{S}, \mathcal{L}, d\}$ zajedno sa mjerom ugla m.

Definicija

U Euklidovoj ravni, Euklidova mjera ugla $\measuredangle ABC$ je

$$m_E(\measuredangle ABC) = \cos^{-1}\left(\frac{\langle A-B, C-B \rangle}{\|A-B\| \cdot \|C-B\|} \right)$$

U Euklidovoj ravni $\mathcal{E}$ izračunati $m_E(\measuredangle ABC)$ ako su $A(0,3)$, $B(0,1)$ i $C(\sqrt{3},2)$.

Rj.

$$\left.\begin{array}{l} A-B=(0,2) \\ C-B=(\sqrt{3},1) \end{array}\right\} \Rightarrow \begin{array}{l} \langle A-B, C-B \rangle = 2 \\ \|A-B\| = \sqrt{0+4} = 2 \\ \|C-B\| = \sqrt{3+1} = 2 \end{array} \Rightarrow \frac{\langle A-B, C-B \rangle}{\|A-B\| \cdot \|C-B\|} = \frac{2}{2 \cdot 2} = \frac{1}{2}$$

$$m_E(\measuredangle ABC) = \cos^{-1}\left(\frac{1}{2}\right) = 60$$

Propozicija m_E je mjera ugla na $\{\mathbb{R}^2, \mathcal{L}_E, d_E\}$

Ovu propoziciju ćemo dokazati u jednoj od sljedećih lekcija.

Definicija (Euklidova tangenta, Euklidova tangentna poluprava)

Ako je $\overrightarrow{BA}$ poluprava u Poincaré-ovoj ravni gdje je $B=(x_B, y_B)$ i $A=(x_A, y_A)$ tada Euklidova tangenta na $\overrightarrow{BA}$ u tački B je

$$T_{AB} = \begin{cases} (0, y_A - y_B), & \text{ako je } \overleftrightarrow{AB} \text{ tip I prave;} \\ (y_B, c - x_B), & \text{ako je } \overleftrightarrow{AB} \text{ tip II prave } {}_cL_r,\ x_B < x_A; \\ -(y_B, c - x_B), & \text{ako je } \overleftrightarrow{AB} \text{ tip II prave } {}_cL_r,\ x_B > x_A. \end{cases}$$

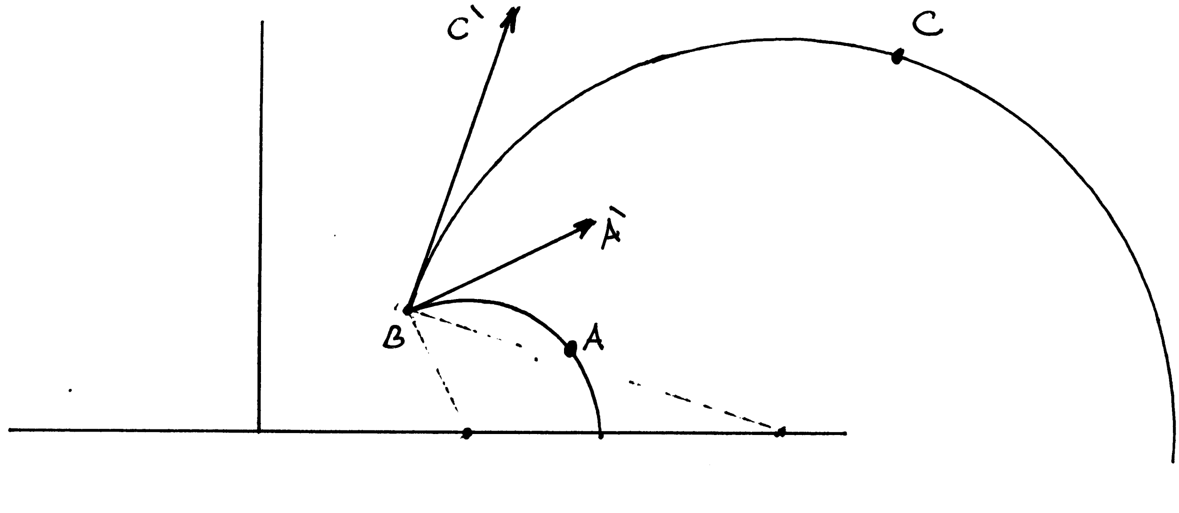
Euklidova tangentna poluprava na $\overrightarrow{BA}$ je Euklidova poluprava $\overrightarrow{BA'}$ gdje je $A' = B + T_{BA}$.

Definicija (Poincaré-ova mjera ugla)

Poincaré-ova mjera ugla $\measuredangle ABC$ u $\mathbb{H}$ je

$$m_H(\measuredangle ABC) = m_E(\measuredangle A'BC') = \cos^{-1}\left(\frac{\langle T_{BA}, T_{BC}\rangle}{\|T_{BA}\| \cdot \|T_{BC}\|}\right) \qquad \ldots(1)$$

gdje su $A' = B + T_{BA}$ i $C' = B + T_{BC}$, a $m_E(\measuredangle A'BC')$ je Euklidova mjera Euklidovog ugla $\measuredangle A'BC'$.



Primjetimo da u jednačini (1) zapravo netrebamo A' i C' da izračunamo $m_H(\measuredangle ABC)$, trebaju nam samo T_{BA} i T_{BC}.

(#) U Poincaré-ovoj ravni izračunati mjeru $\measuredangle ABC$ gdje su $A(0,1)$, $B(0,5)$ i $C(3,4)$.

Rj. Nije teško izračunati da je
$\overleftrightarrow{BC} = {}_0L_5$, $\overleftrightarrow{CA} = {}_4L_{\sqrt{17}}$ i $\overleftrightarrow{BA} = {}_0L$

$B(0,5)$
$C(3,4)$ $\quad x_B < x_C$
$\overleftrightarrow{BC} = {}_0L_5 \quad c=0$

Time je $T_{BA} = (0,-4)$

$T_{BC} = (5,0)$

$m_H(\measuredangle ABC) = \theta$ gdje je

$$\cos\theta = \frac{\langle T_{BA}, T_{BC} \rangle}{\|T_{BA}\| \cdot \|T_{BC}\|} = \frac{0}{20} = 0$$

$\cos(\theta) = 0$ i $m_H(\measuredangle ABC) = 90$.

<u>Propozicija</u> m_H je mjera ugla na $\{\mathbb{H}, \mathcal{L}_H, d_H\}$.

Ovu propoziciju ćemo dokazati u jednoj od sljedećih lekcija.

<u>Dogovor</u> Od sad pa nadalje, termini Euklidova ravan, Poincaré-ova ravan i Taxicab ravan (Taksi ravan) redom će označavati (ćemo se pozivati na) sljedeće protractor geometrije

$$\mathcal{E} = \{\mathbb{R}^2, \mathcal{L}_E, d_E, m_E\},$$
$$\mathcal{H} = \{\mathbb{H}, \mathcal{L}_H, d_H, m_H\},$$
$$\mathcal{T} = \{\mathbb{R}^2, \mathcal{L}_E, d_T, m_E\}.$$